Robust asset allocation under model risk

Financial investors often develop a multitude of models to explain financial securities' dynamics, none of which they can fully trust. Model risk (also referred to as ambiguity) prevents investors from using the classical framework of expected utility maximisation to calculate optimal portfolio allocations. We propose an easily implementable approach to account for model risk in a robust way.

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THIS ARTICLE AIMS to characterise and construct a methodology for robust portfolio allocation under model risk, that is, when investors consider different models to take their allocation decision.

More precisely, let us consider a financial market with N risky assets and a risk-free asset. An investor wants to allocate their wealth among these assets by choosing $\phi \equiv (\phi_0, \ldots, \phi_N)$, the vector of weights for the risk-free asset and the N risky assets.

The standard framework for investment decision-making was developed by Markowitz (1952), where there is no model uncertainty. The optimal portfolio allocation is obtained as:

$$\phi^* \equiv \arg\max_{\phi} \mathbb{E}_{\mathbb{P}}\left[u\left(X^{\phi}, \lambda\right)\right] \qquad (1)$$

where u is a Von-Neumann Morgenstern utility function characterising the investor's preferences and parameterised by the risk-aversion parameter λ, and X^{ϕ} stands for the terminal value of the portfolio at a given time horizon. In this setting, $\mathbb{P}$ stands for the only prior (or model for the distribution of the asset returns) the investor has, and is known without ambiguity. Hence, the investor's risk is perfectly quantifiable through the knowledge of the distribution $\mathbb{P}$.

The investor may however consider different models $\mathbb{Q}$ in a set of possible models $\mathcal{Q}$. In this case, the investment problem is modified according to the subjective view $\pi(\mathbb{Q})$ of the investor on each model $\mathbb{Q}$. More precisely, $\pi(\mathbb{Q})$ represents the subjective likelihood of the model $\mathbb{Q}$ for the investor. The investor operates a linear blending of the different models, weighted by their subjective probability $\pi(\mathbb{Q})$ of being the 'real' model. Under each model, the investor considers the objective expected utility of their future wealth. Across all priors, the investor considers the subjective expected value of the expected utilities under the different models. Such a framework is referred to as subjective expected utility (SEU) and was first introduced by Savage (1954). The portfolio allocation becomes:

$$\phi^* \equiv \arg\max_{\phi} \sum_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\left[u\left(X^{\phi}, \lambda\right)\right]\mu(\mathbb{Q}) \qquad (2)$$

Note that in this framework, even if the investor has several priors as reference, they do not show any aversion towards the co-existence of different models to take their decision. They are neutral towards model uncertainty as there exists no ambiguity about the set of models considered and their likelihood to occur.

However, as shown by Ellsberg (1961), the decision-makers show a more averse behaviour when betting on events for which outcomes are ambiguous (that is, when there is also some uncertainty regarding the underlying model) than when betting on events for which the outcomes are only risky (that is, the underlying model is well known). A financial illustration of the Ellsberg paradox is the risk premium paradox: investors tend to invest more in their local market even though the expected return is lower than for foreign markets. This is because investors add an ambiguity premium to risky assets (investors prefer investing in assets located in their geographical zone, because they feel they can apprehend better their return distribution).

The SEU framework fails to take into account this additional source of aversion for investors. Various approaches have been developed in the literature to take into account this aversion towards model uncertainty in the investment decision process. Among them, Gilboa & Schmeidler (1989) proposed a min-max approach leading to a very conservative decision rule based upon the worst-case model. More recently, Klibanoff, Marinacci & Mukerji (2005) introduced a generalised model. They consider an increasing, concave transformation function Ψ characterising the investor ambiguity aversion through a parameter γ. The optimal weights vector is then determined as:

$$\phi^* \equiv \arg\max_{\phi} \sum_{\mathbb{Q}\in\mathcal{Q}} \Psi\left\{\mathbb{E}_{\mathbb{Q}}\left[u\left(X^{\phi}, \lambda\right)\right], \gamma\right\}\mu(\mathbb{Q}) \qquad (3)$$

The main feature of this model is that it unifies all the previous approaches accounting for model ambiguity. However, this theoretical approach can be very challenging to implement in practice for different reasons, including the calibration of the various parameters. Indeed, no distinction is made between specific ambiguity aversion for a given model ('How good is this specific model to represent the reality?') and general ambiguity aversion for the whole class of models ('How much can all the models explain the reality?'). Moreover, solving explicitly the program (3) can be extremely difficult, even numerically, especially in the multi-dimensional case or when adding some constraints on the portfolio allocation.[1] Finally, such an approach lacks some flexibility in the sense that if the investor considers a new model, they have to recalculate the program entirely. To overcome those limitations, we propose a robust, general framework for decision-making under uncertainty. We are not aiming to find the optimal solution for a given criterion but more at finding a robust solution.

A robust approach to model risk

We propose a new approach to model ambiguity that is more flexible, easier to calculate, more robust and more tractable than the methods proposed in the literature. This ambiguity robust adjustment (ARA) approach is independent of the set of models considered, as well as of the choice criterion to define the optimal portfolio under each model $\mathbb{Q} \in \mathcal{Q}$.

More precisely, we proceed in two steps and introduce a distinction between two types of ambiguity:

- Absolute ambiguity. This refers to the ambiguity the investor has for a given model. We first solve the optimisation problem assuming that the model considered is the true model. We thus calculate a distorted expected value of the deduced optimal weights, transformed by an absolute ambiguity robust adjustment (AARA) function denoted ψ. Note that the absolute adjustment is made on the solution and not on the choice criterion. This allows us some additional flexibility in the use of each model.
- Relative ambiguity. This expresses the relative ambiguity the investor has among their different models. In a second step, we aggregate the adjusted optimal weights calculated for each model through a relative ambiguity robust adjustment (RARA) function, denoted π.

The ARA portfolio allocation is therefore obtained as:

$$\phi^* \equiv \sum_{\mathbb{Q} \in \mathcal{Q}} \psi \left\{ \arg\max_{\phi} \mathbb{E}_{\mathbb{Q}} \left[u\left(X^{\phi}, \lambda \right) \right], \gamma \right\} \pi(\mathbb{Q}) \tag{4}$$

We describe below the characteristics of the functions ψ and π.

- **AARA.** The idea behind the AARA adjustment is that the investor, because they have doubts about the optimal weights generated by a given model, wishes to scale down those weights and especially the biggest absolute weights that could entail the biggest risks in their portfolio.

We used a very similar function to the one used by Klibanoff, Marinacci & Mukerji (2005) to account for ambiguity. The function ψ can be any classical S-shape function that has the property of being concave (convex for negative values), symmetric and monotonic (similar attributes to classical utility functions). What really characterises the function ψ is its ambiguity parameter γ, which accounts for the concavity of the function ψ and therefore the ambiguity aversion of the investor (*see figure 1*).

An example for the function ψ is:

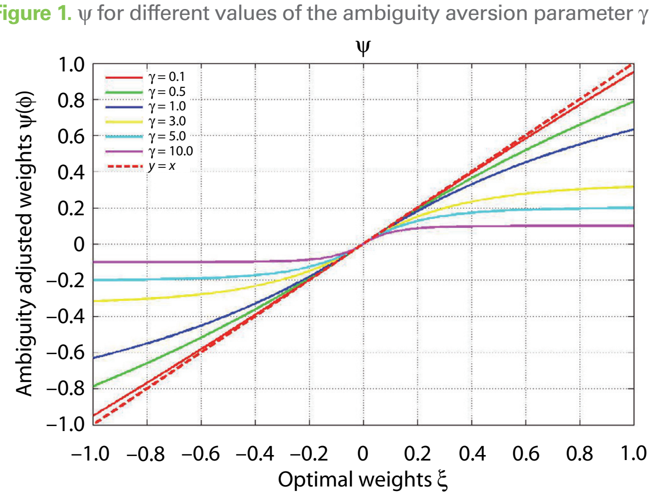
$$\psi(x,\gamma) \equiv \begin{cases} \frac{1-\exp^{-\gamma x}}{\gamma}, 0 \le x \le 1 \\ \frac{\exp^{\gamma x}-1}{\gamma} - 1 \le x \le 0 \end{cases} . \tag{5}$$

- **Role of the risk-free asset.** Due to its specific nature, the risk-free asset has no model risk associated with it (its future value is known with certainty). Therefore, it plays a specific role in the ambiguity adjusted optimal asset allocation. It can be assimilated to a refuge value in the following sense: the more the investor is averse to ambiguity, the more they will invest in the risk-free asset. In this sense, as the 'disinvested' part of the wealth from the risky assets is transferred to the risk-free asset, the adjusted weight of the risk-free asset corresponds to the amount of money the investor is reluctant to invest in risky assets because of their aversion towards model risk.
- **RARA.** Once the optimal solutions have been calculated for each prior $\mathbb{Q}$ and have been independently adjusted for ambiguity aversion through the AARA function ψ, we need to aggregate them across all priors in the set $\mathcal{Q}$. The RARA function takes into account the ambiguity aversion of each prior relative to the whole class of priors $\mathcal{Q}$. Such an adjustment is made through a mixture measure π. The RARA function $\pi(\mathbb{Q})$ represents the likelihood or degree of confidence the decision-maker has in the adjusted result given under $\mathbb{Q}$ when knowing all the adjusted results for all the other priors.

Therefore, $\pi(\mathbb{Q})$ can be seen as a subjective weight given by the decision-maker to the adjusted solution for the model $\mathbb{Q}$. $\pi(\mathbb{Q})$ will therefore always be non-negative. If the decision-maker does not trust the prior $\mathbb{Q}$ relative to the other priors, they will simply set the weight to zero. If, on the contrary, they fully trust the prior $\mathbb{Q}$ relative to the other priors, then the weight for all the other priors will be zero. The total weight $\sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q})$ is not necessarily one, since the investor may still believe that they do not have a full understanding of the situation.

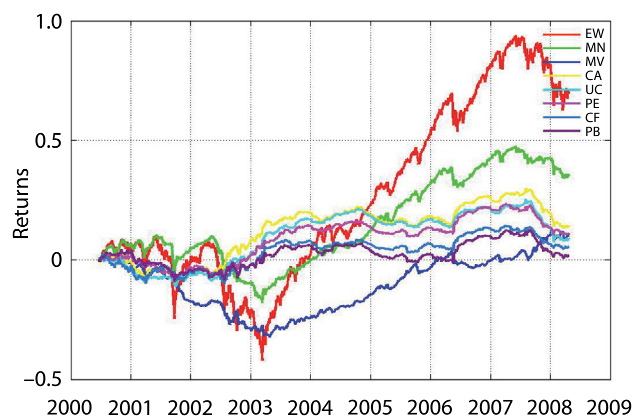
- **ARA parameterisation.** The investor aversion to ambiguity is dynamic, in the sense that depending on the period considered, they will be more or less confident about their models and the overall set of models they consider.

Figure 1. ψ for different values of the ambiguity aversion parameter γ



1 Klibanoff, Marinacci & Mukerji (2005) only give a simple numerical example for a portfolio with three assets, whereas practitioners often consider portfolios with hundreds of assets. We have compared their example with our methodology in Barrieu & Tobelem (2008) and we also provide a more complex theoretical example that can be solved in close form if using our methodology

Figure 2. Returns of strategies



Therefore, we allow the function π and the ambiguity aversion parameter γ to adapt dynamically and expand or contract the total investment size whether the total ambiguity aversion decreases or increases over time (the ambiguity parameter γ and the measure π can be reparameterised at every decision time). As pointed out by Epstein & Schneider (2007), the ambiguity aversion of an investor is not monotonically decreasing over time. Our RARA function allows the investor to adjust their portfolio weights dynamically, depending on their overall belief of how much their models can explain the true distribution $\mathbb{P}$.

The approach is rather different from a classical Bayesian updating approach, where the investor learns more about the underlying model with any new information flowing in the stock price returns. In a Bayesian framework, the investor believes that once they get enough information, they will ultimately converge towards the true model and therefore is gradually and monotonically more and more confident about their model. Under model ambiguity, however, this is not the case. The investor can become more or less confident over time, in a non-monotonic way. They do not assume that more information can systematically give them more confidence about their model.

Many methods could be used in order to calibrate the different measures $\pi(\mathbb{Q})$, as well as the ambiguity aversion parameter $\gamma^{\mathbb{Q}}$ for a given model $\mathbb{Q}$. We propose a simple empirical methodology that takes into account the relative historical performance of the different models. First, we calculate the time series of performance measures on the different models considered, evaluated over a given time window. The measure π can then be calculated as a weighted average of the performance measures, whereas the ambiguity aversion parameter γ can be parameterised as the inverse of those performance measures.

Empirical tests

Here, we detail the results of some tests we run to evaluate the performance of ambiguity robust portfolios, compared with other classical optimised portfolios. The tests were run on European assets. We collected daily closing prices from January 2000–March 2008.[2] Our set of securities is the set of Eurostoxx 600 constituents that were trading across the whole period. For this empirical study, we disregard the risk-free rate, as we consider a portfolio total return, as would be done for strategies without benchmarks such as hedge funds or proprietary groups' strategies. We assume that our transaction costs, fees and slippage correspond to 3 basis points of the daily turnover of the strategy. We run a back test on historical data when at each date the investor rebalances their portfolio by re-estimating the different models and resetting their optimal investment weights. The performances of the different strategies were then evaluated. We find that our ARA approach allows the investor to achieve superior performance compared with that achieved by classically optimised portfolios.

- **Portfolios tested.** We use an estimation window to estimate the following portfolio weights:
- The equally weighted portfolio (EW), which gives an equal weight to all the risky assets.
- The minimum variance portfolio (MN), which is the portfolio with minimum variance.
- The mean variance portfolio (MV), which is effectively the Markowitz portfolio.
- The capital asset pricing model (CAPM) portfolio weights (CA). We base our CAPM portfolio on the Jensen alphas. We estimate the CAPM betas over the estimation window. We calculate the Jensen alpha as the difference between the observed return and the beta-adjusted market return for each risky asset. We then define the CAPM weights as the weighted average alpha across all the risky assets considered.
- The CAPM uncertain portfolio (UC). We define the UC weights as the

2 Equivalent to 2,000 business days

Table A: Performance of strategies against the SXXP index

	EW	MN	MV	CA	UC	PE	CF	PB
μ (%)	69.74	35.03	10.68	13.95	8.71	11.04	5.42	1.89
$\bar{\mu}$ (bp)	3.41	1.71	0.52	0.68	0.43	0.54	0.26	0.09
σ (%)	15.26	6.77	4.75	4.71	3.95	4.38	5.06	4.93
max(μ) (bp)	470.80	199.81	133.63	182.75	210.82	193.83	225.39	234.74
min(μ) (bp)	–540.82	–317.57	–219.94	–162.10	–139.12	–217.76	–205.43	–234.40
Sharpe	0.56	0.63	0.27	0.36	0.27	0.31	0.13	0.05
Sortino	0.72	0.75	0.32	0.52	0.40	0.44	0.19	0.07
Gain/loss (%)	110.48	112.05	104.98	106.77	104.94	105.88	102.48	100.88
Win/lose (%)	119.16	123.96	122.98	101.00	100.60	102.52	99.11	98.72
CER (bp)	2.94	1.62	0.48	0.64	0.39	0.50	0.21	0.04
T/O (%)	2.89	21.85	28.60	135.31	135.01	141.51	140.83	141.59

CAPM weights adjusted by the variance of the CAPM residual.

We also consider three fundamental portfolios based on stock-specific financial ratios:

- The price earning portfolio (PE). We calculate the relative price earning ratio (PER) return of a stock among its sector peers and we give positive weights to the lower PER stocks and negative weights to the higher ones.
- The cashflow portfolio (CF). We compute the relative cashflow ratio (CFR) return of a stock among its sector peers and we give positive weights to the lower CFR stocks and negative weights to the higher ones.
- The price-to-book portfolio (PB). We calculate the relative price-to-book ratio (PBR) return of a stock among its sector peers and we give positive weights to the lower PBR stocks and negative weights to the higher ones.
- The ambiguity robust portfolio (RA). Finally, we define by π the vector of relative ambiguity aversion weights given to the models considered. The optimal ambiguous portfolio is therefore defined as the different models' weights weighted by π.

● **Performance measures.** To parameterise the absolute ambiguity parameter γ of the function ψ and the relative ambiguity function π, we use different portfolio performance measures:

- The Sharpe ratio, which represents the ratio of the mean return of a portfolio over its standard deviation.
- The Sortino price ratio, which represents the ratio of the mean return of a portfolio over the standard deviation of its negative returns.
- The gain/loss ratio, which is the ratio of total positive returns over total negative returns.
- The win/lose ratio, which is similar to the gain/loss ratio. It is the ratio of the number of total positive returns over the number of total negative returns.

The following two measures are used to compare different portfolio performances.

- Certain equivalent. This corresponds to the equivalent risk-free return of the portfolio return (the portfolio return adjusted for its risk-aversion adjusted standard deviation), as used by DeMiguel, Garlappi & Wang (2007) in their comparative study of portfolio performance.
- Turnover. This corresponds to the change in portfolio weights from one rebalancing period to the next. The investor aims to reduce the turnover as trading implies costs (exchange fees, price impact, etc).

As an estimate for π, we choose the relative performance measure of a model within the class of models. Similarly, γ is estimated by the inverse absolute performance measure of a model (the worse the performance, the bigger the ambiguity aversion).[3]

● **Results.** First, we describe the individual performances of the five models considered. Then, we show how the SEU portfolios of Savage (as defined above and where the single strategies are linearly weighted by the weighted average of their different performance measures) outperform the individual strategies.[4] Finally, we display the performances of the ambiguity robust portfolios, parameterised by the four different performance measures considered. We conclude that the ambiguity robust portfolios outperform by far the non-ambiguous SEU portfolios.

- Performances of the different models. The cumulative return of the different models over the period January 2000–March 2008 is plotted in figure 2.

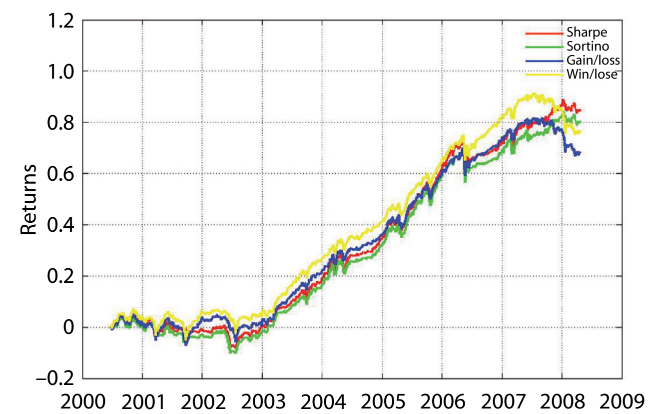
3 Note that if the performance measure of a model is zero, then p and g are defaulted to zero

4 We use the SEU portfolios as benchmarks for our ARA methodology, as the Klibanoff, Marinacci & Mukerji (2005) model is almost impossible to calculate for a large universe of assets

Table B: Performance of SEU strategies against the SXXP index

	Sharpe	Sortino	GainLoss	WinLose
μ (%)	77.59	75.26	55.72	59.58
$\bar{\mu}$ (bp)	3.79	3.68	2.72	2.91
σ (%)	7.06	6.92	6.89	6.95
max(μ) (bp)	326.58	322.47	310.73	321.65
min(μ) (bp)	–411.38	–412.40	–275.82	–270.02
Sharpe	1.34	1.33	0.99	1.05
Sortino	1.52	1.50	1.25	1.33
Gain/loss (%)	128.85	128.59	119.60	120.95
Win/lose (%)	135.45	135.09	121.06	123.23
CER (bp)	3.69	3.58	2.63	2.81
T/O (%)	76.33	78.26	112.96	112.65

Figure 3. Returns of robust ambiguity strategies



We have calculated in table A the statistics of the different strategies, for the whole period considered and annualised. The best strategy in terms of almost all performance measures (Sharpe, Sortino, gain/loss ratio and win/lose ratio) is the MN portfolio.

We use the four performance measures described previously to parameterise π and γ and construct our ambiguity robust portfolios. At each date t, we calculate the performance measure of the different strategies over the historical returns between date $t - 120$ and $t - 1$. Even though the UC and CA strategies' returns underperform the EW strategy return over the whole period considered (the equally weighted portfolio has the biggest total return, of almost 70%), there exist some periods where the reverse is true (in 2002, for instance, the CA and UC portfolios perform better). This confirms that the relative weight of each models should be dynamic, as proposed in our ambiguity robust methodology.[5]

- SEU portfolio. In table B, we show the performances and statistics for the SEU portfolios, which weight the different models linearly with the measure π (respectively calculated with the four different performance measures Sharpe, Sortino, gain/loss and win/lose ratios), without considering the

5 To have numbers of similar magnitude, we scale the inverse Sharpe and Sortino measures by a reference ratio of three annualised. In practice, it means that an investor considers a portfolio for which the return is three times as big as its risk as benchmark. We have also capped the performance measures by three to prevent a strongly performing model dominating the others. Formally, we have: Sharpe = min(max(0, 3/Sharpe, 3)) and Sortino = min(max(0, 3/Sortino, 3))

Table C: Performance of ambiguity robust strategies against the SXXP index

	Sharpe	Sortino	GainLoss	WinLose
μ (%)	84.79	80.28	68.18	76.39
$\bar{\mu}$ (bp)	4.14	3.92	3.33	3.73
σ (%)	6.97	6.78	6.79	7.06
max(μ) (bp)	351.97	344.72	300.82	321.42
min(μ) (bp)	–423.97	–422.49	–276.14	–282.74
Sharpe	1.49	1.45	1.23	1.32
Sortino	1.64	1.58	1.51	1.65
Gain/loss (%)	133.65	132.56	124.66	126.87
Win/lose (%)	137.44	137.94	128.97	130.00
CER (bp)	4.05	3.83	3.24	3.63
T/O (%)	67.21	70.41	98.71	98.49
RAA	0.31	0.30	0.43	0.45
AAA	1.61	1.57	1.86	1.90

absolute ambiguity adjustment from the function ψ. The SEU portfolios outperform almost all the single strategy portfolios. The SEU portfolios outperform the individual strategies. However, we improve greatly these performances with our ambiguity robust approach as we show below.

- Ambiguity robust portfolios. In figure 3, we have plotted the performance of the four different ambiguity robust portfolios (where the ambiguity parameters γ and π are estimated using the four different performance measures: Sharpe, Sortino, gain/loss and win/lose ratios). In table C, we have calculated the statistics of the ambiguity robust portfolio estimated with the γ and π measures generated by the four different performance measures. The four portfolios generated outperformed the best CA strategy portfolio in terms of all four different performance measures.

The absolute ambiguity aversion (AAA) and the relative ambiguity aversion (RAA) measures allow us to quantify our ambiguity adjustment.[6] The biggest ambiguity adjustments are made for the worst-performing strategies (based on win/lose and gain/loss ratios). As we can see, the ambiguity robust portfolios outperform all the SEU portfolios, meaning that the ψ adjustment enhances the performance of ambiguity-averse investors' portfolios.

Conclusion

The aim of our article is to provide a simple and easy to implement methodology for practitioners who want to allocate optimally and in a robust way their portfolio of assets when they have several models for the asset returns distributions but are ambiguous about each of them (that is, they do not fully trust them). We propose a simple, robust and systematic method of combining the different weights calculated conditionally for each model. The concrete methodology we develop enables us to account for ambiguity via practical empirical measures, such as the performance measures of each conditional portfolio.

Our method is very different from that of Klibanoff, Marinacci & Mukerji (2005) as we do not proceed to a unique complex optimisation that is challenging to solve in practice. Our approach is more practical and industry-orientated: practitioners can easily define the optimal weights for each of the models they consider, then they can mix those prior conditional weights, taking into account the absolute and relative ambiguity they have against each model, as reflected by the concavity of ψ and the weight $\pi(\mathbb{Q})$ attributed to each model $\mathbb{Q}$.

The parameterisation we propose for the AAA and the RAA in our empirical example is by no means optimal. However, we have shown that it can greatly enhance the performance of some classical portfolio strategies. The proposed robust ambiguity portfolio is more stable, with a lower turnover, and less risky, with a higher certain equivalent ratio, than unadjusted portfolios as well as SEU portfolios. The ARA methodology smooths and reduces the risk of classical strategies. One of the main features of the ARA methodology is to provide a dynamic adjustment for model uncertainty. As the parameterisation depends on the past performance of the strategy, it would be interesting to study how the ARA model would behave in major downturns, and adapt to different market regimes. L&P

6 We define the value of the absolute ambiguity aversion (AAA) of an investor toward the model $\mathbb{Q}$ as the theoretical turnover to rebalance the investor asset allocation from the optimal weights obtained if the prior $\mathbb{Q}$ is assumed to be the real model $\mathbb{P}$ to the absolute ambiguity adjusted portfolio, taking into account the investor absolute aversion against their prior $\mathbb{Q}$. We define the value of the relative ambiguity aversion (RAA) of an investor as the turnover between the robust ambiguity portfolio and the subjective expected utility portfolio

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